

Nonabelian KP hierarchy with Moyal algebraic coefficients

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Abstract

A higher dimensional analogue of the KP hierarchy is presented. Fundamental constituents of the theory are pseudo-differential operators with Moyal algebraic coefficients. The new hierarchy can be interpreted as large- N limit of multi-component ($\mathfrak{gl}(N)$ symmetric) KP hierarchies. Actually, two different hierarchies are constructed. The first hierarchy consists of commuting flows and may be thought of as a straightforward extension of the ordinary and multi-component KP hierarchies. The second one is a hierarchy of noncommuting flows, and related to Moyal algebraic deformations of selfdual gravity. Both hierarchies turn out to possess quasi-classical limit, replacing Moyal algebraic structures by Poisson algebraic structures. The language of W -infinity algebras provides a unified point of view to these results.

Key words: KP hierarchy, Moyal algebra, higher dimensional integrable systems,
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1. Introduction

The technique of large- N limit is a magic that produces new continuous dimensions out of finite matrix indices. According to observations since the early eighties [1] the large- N limit of $\mathfrak{sl}(N)$ looks like the algebra of infinitesimal area-preserving (or symplectic) diffeomorphisms on a two dimensional surface Σ :

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$$\lim_{N \rightarrow \infty} \mathfrak{sl}(N) \simeq \text{sdiff}(\Sigma). \quad (1.1)$$

If central elements (scalar matrices) are added, the limit becomes a Poisson algebra, i.e., the Lie algebra realized by a Poisson bracket on the surface:

$$\lim_{N \rightarrow \infty} \mathfrak{gl}(N) \simeq \text{Poisson}(\Sigma). \quad (1.2)$$

These intriguing observations are based on the existence of the so called “sine generators” in these matrix Lie algebras. The theory of sine generators further tells us [1] that this is just a special, rather singular case; in general, the limit can also become a Moyal algebra:

$$\lim_{N \rightarrow \infty} \mathfrak{gl}(N) \simeq \text{Moyal}(\Sigma). \quad (1.3)$$

Moyal algebras are a kind of “quantum deformation” of Poisson algebras, which is a unique deformation subject to a set of natural requirements [2].

These observations, although mathematically slightly problematical [3] have been a very useful point of view for understanding higher dimensional integrable systems. For instance, the large- N limit of the $\mathfrak{sl}(N)$ Toda field theory becomes a three dimensional nonlinear system (the dispersionless Toda equation) [4]. Similarly, the large- N limit of two dimensional nonlinear sigma models with the pure WZW Lagrangian coincides with selfdual gravity (the selfdual vacuum Einstein equation) [5]. These two models are related to Poisson algebras, and well known to be integrable. Recently, a Moyal algebraic analogue of selfdual gravity was discovered from the same standpoint [6] and shown to be integrable [7].

Inspired by this point of view, we present in this paper a higher dimensional analogue of the KP hierarchy. The celebrated KP hierarchy is known to possess a $\mathfrak{gl}(N)$ symmetric version, the so called “ N -component KP hierarchy” [8]. Our new hierarchy may be thought of as a kind of large- N limit of the N -component KP hierarchy. Actually, rather than literally considering such limit, we directly construct the hierarchy replacing $\mathfrak{gl}(N)$ by the Moyal algebra $\text{Moyal}(M)$ on a symplectic manifold M . For simplicity, we mostly deal with the planar case (i.e., the case where M is a two dimensional symplectic manifold), but all results can be extended straightforwardly to more general cases.

The new hierarchy (which we call the “nonabelian KP hierarchy with Moyal algebraic coefficients”) turns out to possess several novel characteristics. Of particular interest is the fact that there are two distinct types of hierarchies within this framework. The first type of hierarchy consists of a commuting set of flows like many other integrable hierarchies. This hierarchy may be thought of as the naive large- N limit of the N -component KP hierarchy. The second type, meanwhile, is a hierarchy of noncommuting flows. This noncommuting hierarchy has a reduction to a hierarchy of integrable flows including Moyal

algebraic deformations of selfdual gravity. These Moyal algebraic deformations are somewhat distinct from Strachan's deformation.

Another large- N limit, i.e., the Poisson algebraic limit, can be realized as quasi-classical ($\hbar \rightarrow 0$) limit of these Moyal algebraic hierarchies. The quasi-classical limit of the commuting hierarchy gives a higher dimensional extension of the dispersionless KP hierarchy [9]. Similarly, the quasi-classical limit of the noncommuting hierarchy is related to hierarchies of integrable flows in ordinary selfdual gravity [10].

This paper is organized as follows. Sections 2 and 3 are intended to be a review of key ideas and tools in the theory of integrable hierarchies and Moyal algebras. The contents of these sections, in particular Section 2, are mostly model independent. Sections 4 and 5 are specialized to the Moyal algebraic hierarchies. The relation to Moyal algebraic deformations of selfdual gravity is discussed in Section 6. The quasi-classical limit is treated in Section 7. Section 8 is a summary of results from the point of view of W -infinity algebras.

2. General structure of integrable hierarchies

We here present a general framework for dealing with various integrable hierarchies on an equal footing. A prototype is the standard treatment of the ordinary and multi-component KP hierarchies [8], which is now reformulated in an abstract and model-independent way.

2.1. Lax and zero-curvature representations

Integrable hierarchies usually possess two distinct representations—the Lax representation and the zero-curvature (or Zakharov–Shabat) representation. Algebraic structures of these representations are governed by a Lie algebra (of operators or of matrices) with a direct sum decomposition into two subalgebras:

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-. \quad (2.1)$$

Let $(\)_{\pm}$ denote the projection onto each component.

Lax and zero-curvature representations consist of differential equations (“Lax equations” and “zero-curvature equations”) for t -dependent elements L_{σ} and B_i of the above Lie algebras:

$$L_{\sigma} = L_{\sigma}(t) \in \mathcal{G}, \quad B_i = B_i(t) \in \mathcal{G}_+, \quad (2.2)$$

where σ ranges over a finite index set, i over another (finite or infinite) index set. Let us call the L_{σ} 's “Lax operators” and the B_i 's “Zakharov–Shabat operators.” The Lax representation can be written

$$\frac{\partial L_{\sigma}}{\partial t_i} = [B_i, L_{\sigma}], \quad (2.3)$$

where t_i denotes a time variable of the “ i -th” flow. The zero-curvature equations (or “Zakharov–Shabat equations”) can similarly be written

$$\frac{\partial B_i}{\partial t_j} - \frac{\partial B_j}{\partial t_i} + [B_i, B_j] = 0. \tag{2.4}$$

The Lax and Zakharov–Shabat operators should be selected so that these equations be consistent.

Example (KP hierarchy): For the KP hierarchy, \mathcal{G} is comprised of pseudo-differential operators (of finite order) in one variable x with scalar coefficients,

$$\begin{aligned} \mathcal{G} &= \{A \mid A = \sum a_n \partial_x^n\}, \quad \partial_x = \partial / \partial x, \\ \left(\sum a_n \partial_x^n\right)_+ &= \left(\sum a_n \partial_x^n\right)_{\geq 0} = \sum_{n \geq 0} a_n \partial_x^n, \\ \left(\sum a_n \partial_x^n\right)_- &= \left(\sum a_n \partial_x^n\right)_{\leq -1} = \sum_{n \leq -1} a_n \partial_x^n. \end{aligned} \tag{2.5}$$

The Lax and zero-curvature representations are formulated in terms of a single Lax operator L of the form

$$L = \partial_x + \sum_{n=1}^{\infty} g_{n+1} \partial_x^{-n} \tag{2.6}$$

and an infinite number of Zakharov–Shabat operators B_n , $n = 1, 2, \dots$ given by

$$B_n = (L^n)_{\geq 0}. \tag{2.7}$$

Example (multi-component KP hierarchy): The formulation of the N -component KP hierarchy is very similar, but based on pseudo-differential operators with $N \times N$ matrix-valued coefficients. Besides an analogue L of the Lax operator in the one-component KP hierarchy, we need N additional Lax operators U_α , $\alpha = 1, \dots, N$ to formulate the Lax and zero-curvature representations. Flows are numbered by a double index $i = (n, \alpha)$ ($n = 1, 2, \dots$, $\alpha = 1, \dots, N$) and generated by the Zakharov–Shabat operators

$$B_{n\alpha} = (L^n U_\alpha)_{\geq 0}. \tag{2.8}$$

2.2. Dressing operator

Integrable hierarchies of the above type allow another fundamental representation—the dressing operator representation. The dressing operator is a t -dependent group element

$$W = W(t) \in \exp \mathcal{G}_- \tag{2.9}$$

of the subalgebra \mathcal{G}_- with which the Lax operators L_σ are expressed in the “dressing” form

$$L_\sigma = WC_\sigma W^{-1}, \quad (2.10)$$

where the C_α are “undressed” operators that are independent of the solutions under consideration. With a suitable choice of the dressing operator, the Lax and zero-curvature equations can be converted into a set of differential equations of the form

$$\frac{\partial W}{\partial t_i} = B_i W - W G_i, \quad (2.11)$$

where the G_i are a commuting set of operators that are also independent of solutions of the hierarchy. The Zakharov–Shabat operators turn out to be written

$$B_i = \left(W G_i W^{-1} \right)_+, \quad (2.12)$$

so that one can eliminate B_i from the above equations. This results in the equations

$$\frac{\partial W}{\partial t_i} = - \left(W G_i W^{-1} \right)_- W. \quad (2.13)$$

The last equations give a well defined set of flows on the group $\exp \mathcal{G}_-$ generated by \mathcal{G}_- .

Example (KP hierarchy): In the case of the one- and multi-component KP hierarchy, W is a pseudo-differential operator of the form

$$W = 1 + \sum_{n=1}^{\infty} w_n \partial_x^{-n}. \quad (2.14)$$

Remark: In the traditional theory of integrable hierarchies, C_σ 's are usually t -independent. We shall, however, see in later sections that this is too restrictive. Furthermore, we shall extend the present setting to a case where G_i 's are noncommutative.

2.3. Factorization relation

The above flows on $\exp \mathcal{G}_-$ can be translated into a kind of “factorization relation” in the group $\exp \mathcal{G}$ generated by \mathcal{G} (also called a “Riemann–Hilbert problem”). The following formulation is inspired by Mulase’s work on the KP hierarchy [11]. [Remark: In general, such a group $\exp \mathcal{G}$ might not exist in a mathematically rigorous sense, but some realization as a “formal group” is always available as Mulase illustrates for the case of the KP hierarchy. Because of this, we use this somewhat loose notation and dare to call it a “group”.]

A fundamental factorization relation can be written

$$W(t)E(t)W(0)^{-1} = \hat{W}(t) \in \exp \mathcal{G}_+, \tag{2.15}$$

where $W(0)$ is the “initial value” of $W = W(t)$ at $t = 0$, $\hat{W}(t)$ is a t -dependent element of $\exp \mathcal{G}_+$, and $E(t)$ is given by

$$E(t) = \exp \left(\sum t_i G_i \right). \tag{2.16}$$

(To be more precise, one has to enlarge \mathcal{G}_+ into a “formal completion” with respect to time variables [11].) The above relation can be rewritten

$$E(t)W(0)^{-1} = W(t)^{-1}\hat{W}(t), \tag{2.17}$$

which looks more like a “factorization” or a “Riemann–Hilbert problem.” The direct sum decomposition at the level of the Lie algebra, (2.1), yields the direct product decomposition

$$\exp \mathcal{G} \simeq \exp \mathcal{G}_- \times \exp \mathcal{G}_+ \tag{2.18}$$

at the group level. The factorizability is thus ensured at least in a neighborhood of $t = 0$. Eventually, the flows on the space of dressing operators are converted into the action of $E(t)$ on the coset space

$$\begin{aligned} \exp \mathcal{G}_- &\simeq \exp \mathcal{G} / \exp \mathcal{G}_+ \\ W &\mapsto [W] = W \exp \mathcal{G}_+ \end{aligned} \tag{2.19}$$

Let us briefly show how to prove the equivalence of (2.11) and (2.15). (For more details, see Mulase’s paper [11].)

(2.15) \Rightarrow (2.11): Differentiating both sides of (2.15) and recalling (2.16), one can derive the relation

$$\left(\frac{\partial W(t)}{\partial t_i} + W(t)G_i \right) E(t)W(0)^{-1} = \frac{\partial \hat{W}(t)}{\partial t_i}. \tag{2.20}$$

Using the factorization relation once again, one can eliminate the initial data $W(0)$ to obtain

$$\left(\frac{\partial W(t)}{\partial t_i} + W(t)G_i \right) W(t)^{-1} = \frac{\partial \hat{W}(t)}{\partial t_i} \hat{W}(t)^{-1}. \tag{2.21}$$

Since the right hand side gives an element of \mathcal{G}_+ , the \mathcal{G}_- component of the left hand side should be identically zero. This gives Eq. (2.11).

(2.11) \Rightarrow (2.15): Eq. (2.11) can be rewritten

$$\frac{\partial}{\partial t_i} (W(t)E(t)) = B_i(t)W(t)E(t), \tag{2.22}$$

and repeated use of this formula yields a similar expression for higher derivatives,

$$\frac{\partial}{\partial t_{i_1}} \cdots \frac{\partial}{\partial t_{i_k}} (W(t)E(t)) = B_{i_1 \dots i_k}(t)W(t)E(t), \quad (2.23)$$

where $B_{i_1 \dots i_k}$ lies in \mathcal{G}_+ . Evaluated at $t = 0$, they give Taylor coefficients of $W(t)E(t)$,

$$\begin{aligned} W(t)E(t) &= W(0) + \sum_i t_i B_i(0)W(0) + \cdots \\ &= \left(1 + \sum_i t_i B_i(0) + \cdots \right) W(0), \end{aligned} \quad (2.24)$$

and the first factor $1 + \sum_i t_i B_i(0) + \cdots$ on the right hand side becomes an element of $\exp \mathcal{G}_+$. Writing this factor $\tilde{W}(t)$, one finds that the last relation is nothing but the factorization relation.

3. Pseudo-differential operators with Moyal algebraic coefficients

Basic constituents of our new hierarchies, too, are pseudo-differential operators. Pseudo-differential operators in the N -component KP hierarchy have $N \times N$ matrix-valued coefficients. We replace these coefficients by elements of a Moyal algebra $\text{Moyal}(M)$. For simplicity, we here deal with the simplest case where M is a two dimensional plane.

3.1. Moyal bracket and star product

Let (y, z) be a pair of canonical coordinates on a two dimensional planar symplectic manifold. The Moyal algebra consists of functions (or of formal power series) of (y, z) equipped with the Moyal bracket

$$\{a, b\}_{\hbar} = \frac{2}{\hbar} \sinh \left[\frac{\hbar}{2} \left(\frac{\partial^2}{\partial y \partial z'} - \frac{\partial^2}{\partial z \partial y'} \right) \right] a(y, z) b(y', z') \Big|_{y'=y, z'=z}, \quad (3.1)$$

where \hbar is a nonzero parameter (“Planck constant”). Actually, we rather wish to consider the Moyal algebra of formal power series of (y, z) (and of \hbar). This is indeed possible, because the right hand side of the above formula is still meaningful in such a case. The “Planck constant” \hbar then plays only a formal role. [Remark: The above definition of the Moyal bracket is somewhat unusual; usually, “sinh” is replaced by “sin”. This is just for simplifying notations, and one can go back to the ordinary situation by replacing $\hbar \rightarrow i\hbar$.]

The Moyal bracket is a kind of “quantum deformation” of the Poisson bracket

$$\{a, b\} = \frac{\partial a}{\partial y} \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial b}{\partial y}, \tag{3.2}$$

reducing to the latter in the classical limit,

$$\{a, b\}_{\hbar} \longrightarrow \{a, b\} \quad (\hbar \rightarrow 0). \tag{3.3}$$

The canonical coordinates (y, z) are indeed canonical for both these Lie brackets:

$$\{y, z\}_{\hbar} = 1, \quad \{y, z\} = 1. \tag{3.4}$$

Unlike the Poisson bracket, however, the Moyal bracket can be written as a (normalized) commutator

$$\{a, b\}_{\hbar} = \hbar^{-1} (a * b - b * a) \tag{3.5}$$

of an associative product—the star product [1,2]

$$a * b = \exp \left[\frac{\hbar}{2} \left(\frac{\partial^2}{\partial y \partial z'} - \frac{\partial^2}{\partial z \partial y'} \right) \right] a(y, z) b(y', z') \Big|_{y'=y, z'=z}. \tag{3.6}$$

The star product is nothing but the composition rule of Weyl-ordered operators.

3.2. Pseudo-differential operators with Moyal algebraic coefficients

Pseudo-differential operators with coefficients in the Moyal (or star product) algebra are, by definition, linear combinations

$$A = \sum_{n=-\infty}^m a_n \partial_x^n \tag{3.7}$$

of powers of ∂_x with x -dependent coefficients taken from the Moyal algebra:

$$a_n = a_n(\hbar, x, y, z). \tag{3.8}$$

Multiplication of two pseudo-differential operators, which we write $A * B$ and consider an extension of the star product, is defined term-by-term as:

$$(a \partial_x^m) * (b \partial_x^n) = \sum_{k=0}^{\infty} \binom{n}{k} a * \frac{\partial^k b}{\partial x^k} \partial_x^{m+n-k}. \tag{3.9}$$

Actually, it is more natural to expand pseudo-differential operators in powers of $\hbar \partial_x$ rather than ∂_x . This indeed becomes crucial in the construction of integrable hierarchies, as we now show.

We now specify a Lie algebra \mathcal{G} with direct sum decomposition into two subalgebras for our new hierarchies. The Lie algebra \mathcal{G} , by definition, consists of pseudo-differential operators of the form

$$A = \hbar^{-1} \sum_{n=-\infty}^m a_n (\hbar \partial_x)^n \tag{3.10}$$

with Moyal algebraic coefficients that behave smoothly as $\hbar \rightarrow 0$:

$$a_n = a_n^{(0)}(x, y, z) + O(\hbar) \quad (\hbar \rightarrow 0). \tag{3.11}$$

It is not hard to check that these pseudo-differential operators are indeed closed under star product commutator $[A, B] = A * B - B * A$. The direct sum decomposition is basically the same as in the multi-component KP hierarchies:

$$\begin{aligned} \mathcal{G} &= \mathcal{G}_{\geq 0} \oplus \mathcal{G}_{\leq -1}, \\ (\)_+ &= (\)_{\geq 0} : \text{projection onto } (\hbar\partial_x)^0, (\hbar\partial_x)^1, \dots, \\ (\)_- &= (\)_{\leq -1} : \text{projection onto } (\hbar\partial_x)^{-1}, (\hbar\partial_x)^{-2}, \dots \end{aligned} \tag{3.12}$$

The dressing operator should be a group element of $\mathcal{G}_{\leq -1}$:

$$W = \exp_* \hbar^{-1} A, \quad A \in \mathcal{G}_{\leq -1}. \tag{3.13}$$

One can further expand W into negative powers of $\hbar\partial_x$,

$$W = 1 + \sum_{n=1}^{\infty} w_n (\hbar\partial_x)^{-n}, \tag{3.14}$$

but the coefficients w_n will have singularities at $\hbar = 0$ that are rather hard to control. This is by no means a special property of the present situation, but common to this kind of formulation of \hbar -dependent integrable hierarchies.

4. Hierarchy of commuting flows

Having the above Lie algebra of pseudo-differential operators with Moyal algebraic coefficients, we can now proceed to the construction of the first hierarchy that consists of commuting flows. The construction is almost parallel to the ordinary and multi-component KP hierarchies.

4.1. Equations of the dressing operator

Let us assume the setting in the end of the last section, and formulate the hierarchy in the language of dressing operators. As in the case of the N -component KP hierarchy, the index i of flows is a double index (n, α) . Both components, however, range over all nonnegative integers as $n = 0, 1, 2, \dots$, $\alpha = 0, 1, 2, \dots$. The generators G_i of flows are given by

$$G_{n\alpha} = (\hbar\partial_x)^n y^\alpha, \tag{4.1}$$

which obviously commute with each other. The differential equations of the dressing operator are given by

$$\begin{aligned} \hbar \frac{\partial W}{\partial t_{n\alpha}} &= B_{n\alpha} * W - W * G_{n\alpha}, \\ B_{n\alpha} &= \left(W * G_{n\alpha} * W^{-1} \right)_{\geq 0}. \end{aligned} \tag{4.2}$$

To see that these equations fall into the general framework of the previous sections, let us rewrite these equations as

$$\begin{aligned} \frac{\partial W}{\partial t_{n\alpha}} &= \hbar^{-1} B_{n\alpha} * W - W * \hbar^{-1} G_{n\alpha}, \\ \hbar^{-1} B_{n\alpha} &= \left(W * \hbar^{-1} G_{n\alpha} * W^{-1} \right)_{\geq 0}. \end{aligned} \tag{4.3}$$

Now $\hbar^{-1} G_{n\alpha}$ is an element of \mathcal{G} . Since W is assumed to be in $\exp \mathcal{G}_{\leq -1}$, the conjugation $W * \hbar^{-1} G_{n\alpha} * W^{-1}$ again belongs to \mathcal{G} . Therefore $\hbar^{-1} B_{n\alpha}$ becomes an element of $\mathcal{G}_{\geq 0}$. Thus the consistency of the above equations is ensured.

These flows are highly nonlinear and nontrivial in general, but lowest flows contain trivial ones. For instance, the Zakharov–Shabat operators for $(n, \alpha) = (0, 0), (1, 0), (0, 1)$ are given by

$$B_{00} = 1, \quad B_{10} = \hbar \partial_x, \quad B_{01} = y. \tag{4.4}$$

Therefore

$$\frac{\partial W}{\partial t_{00}} = 0, \quad \frac{\partial W}{\partial t_{10}} = \frac{\partial W}{\partial x}, \quad \frac{\partial W}{\partial t_{01}} = \frac{\partial W}{\partial z}, \tag{4.5}$$

i.e., W is independent of t_{00} , and the flows of t_{10} and t_{01} are just translations in x and z .

The factorization relation, too, can be formulated in the same form. The exponential operator $E(t)$ is now given by

$$E(t) = \exp_* \hbar^{-1} \left(\sum_{n,\alpha} t_{n\alpha} (\hbar \partial_x)^n y^\alpha \right), \tag{4.6}$$

where \exp_* denotes the star exponential,

$$\exp_* A = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} A * \dots * A \quad (n\text{-fold star product}). \tag{4.7}$$

With this star exponential operator, the flows on the space $\exp \mathcal{G}_{\leq -1}$ of dressing operators can be identified with the action of $E(t)$ on the coset space $\exp \mathcal{G} / \exp \mathcal{G}_{\geq 0}$.

4.2. Lax and zero-curvature equations

We now consider the operators

$$L = W * \hbar \partial_x * W^{-1}, \quad U = W * y * W^{-1} \tag{4.8}$$

as analogues of the Lax operators L and U_α of the N -component KP hierarchy. They can be written

$$L = \hbar \partial_z + \sum_{n=1}^{\infty} g_{n+1} (\hbar \partial_x)^{-n}, \quad U = y + \sum_{n=1}^{\infty} u_n * L^{-n}, \tag{4.9}$$

and the coefficients turn out to have a smooth limit as $\hbar \rightarrow 0$,

$$g_n = g_n^{(0)}(t, x, y, z) + O(\hbar), \quad u_n = u_n^{(0)}(t, x, y, z) + O(\hbar). \tag{4.10}$$

[Proof: The dressing operator W is assumed to be the star product exponential of an element of \mathcal{G}_{-1} . Evaluating $W * \hbar \partial_x * W^{-1}$ and $W * y * W^{-1}$ by use of the general formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2} [X, [X, Y]] + \dots, \tag{4.11}$$

one can indeed prove the above fact. Q.E.D.]

Zakharov–Shabat operators are given by

$$B_{n\alpha} = (L^n * U^\alpha)_{\geq 0}, \tag{4.12}$$

where we abbreviate $L^n = L * \dots * L$ (n -fold star product) and $U^\alpha = U * \dots * U$ (α -fold star product).

As a consequence of (4.3), these operators turn out to satisfy the Lax equations

$$\hbar \frac{\partial L}{\partial t_{n\alpha}} = [B_{n\alpha}, L], \quad \hbar \frac{\partial U}{\partial t_{n\alpha}} = [B_{n\alpha}, U] \tag{4.13}$$

and the zero-curvature equations

$$\hbar \frac{\partial B_{m\alpha}}{\partial t_{n\beta}} - \hbar \frac{\partial B_{n\beta}}{\partial t_{m\alpha}} + [B_{m\alpha}, B_{n\beta}] = 0 \tag{4.14}$$

with respect to the star product commutator $[A, B] = A * B - B * A$.

4.3. Extended Lax equations

We now introduce another pair of Lax operators. This is inspired by the treatment of quasi-classical limit of the KP hierarchy [12]. The extra Lax operators are given by

$$\begin{aligned} M &= W * E(t) * x * E(t)^{-1} * W^{-1}, \\ V &= W * E(t) * z * E(t)^{-1} * W^{-1}, \end{aligned} \tag{4.15}$$

and satisfy the same Lax equations

$$\hbar \frac{\partial M}{\partial t_{n\alpha}} = [B_{n\alpha}, M], \quad \hbar \frac{\partial V}{\partial t_{n\alpha}} = [B_{n\alpha}, V] \tag{4.16}$$

as well as the canonical commutation relations

$$\begin{aligned}
 [L, M] &= \hbar, & [U, V] &= \hbar, \\
 \text{other commutators} &= 0.
 \end{aligned}
 \tag{4.17}$$

Let us specify the structure of these extra Lax operators in more detail. By use of formula (4.11), one can calculate the “pre-dressing” by $E(t)$:

$$\begin{aligned}
 E(t) * x * E(t)^{-1} &= \sum_{n,\alpha} n t_{n\alpha} (\hbar \partial_x)^{n-1} y^\alpha + x, \\
 E(t) * z * E(t)^{-1} &= \sum_{n,\alpha} \alpha t_{n\alpha} (\hbar \partial_x)^n y^{\alpha-1} + z.
 \end{aligned}
 \tag{4.18}$$

Dressing by W can easily be evaluated by the formulas

$$\begin{aligned}
 W * x * W^{-1} &= x + (\text{element of } \mathcal{G}_{\leq -1}), \\
 W * z * W^{-1} &= z + (\text{element of } \mathcal{G}_{\leq -1}), \\
 W * (\hbar \partial_x)^n y^\alpha * W^{-1} &= L^n * U^\alpha.
 \end{aligned}
 \tag{4.19}$$

Thus, eventually, M and V turn out to be written

$$\begin{aligned}
 M &= \sum_{n,\alpha} n t_{n\alpha} L^{n-1} * U^\alpha + x + \sum_{n=1}^{\infty} h_{n+1} * L^{-n-1}, \\
 V &= \sum_{n,\alpha} \alpha t_{n\alpha} L^n * U^{\alpha-1} + z + \sum_{n=1}^{\infty} v_n * L^{-n},
 \end{aligned}
 \tag{4.20}$$

and the coefficients h_n and v_n , like those of of L and U , have a smooth limit as $\hbar \rightarrow 0$:

$$h_n = h_n^{(0)}(t, x, y, z) + O(\hbar), \quad v_n = v_n^{(0)}(t, x, y, z) + O(\hbar).
 \tag{4.21}$$

Remark: We could have defined $M = W * x * W^{-1}$ and $V = W * z * W^{-1}$, but then extra terms emerge on the right hand side of the Lax equations. Exactly the same phenomenon takes place in the KP hierarchy [12]; the above definition of M and V mimics the formulation developed therein. In this respect, we should define L and U , too, via the same “pre-dressing” by $E(t)$. This however gives the same result because $\hbar \partial_x$ and y commute with $E(t)$.

5. Hierarchy of noncommuting flows

By “noncommuting flows” we mean a set of flows with generators G_i that do not necessarily commute with each other. In general, such noncommutativity causes serious trouble in formulating a Lax formalism. In the following case, however, the generators obey rather special commutation relations, so that we can write down Lax equations etc. in almost the same explicit way as in the case of commuting flows.

5.1. Equation of the dressing operator

The second hierarchy consists of flows with three series of time variables, $t = (t_1, t_2, \dots)$, $p = (p_1, p_2, \dots)$, and $q = (q_1, q_2, \dots)$. The flows are generated by the left action of the star product exponential

$$E(t, p, q) = \exp_* \hbar^{-1} \left(\sum_{n=1}^{\infty} t_n (\hbar \partial)^n - \sum_{n=1}^{\infty} p_n z (\hbar \partial_x)^n + \sum_{n=1}^{\infty} q_n y (\hbar \partial_x)^n \right) \quad (5.1)$$

on the coset space $\exp \mathcal{G} / \exp \mathcal{G}_{\geq 0}$. These flows can be converted into flows on the space $\exp \mathcal{G}_{\leq -1}$ of dressing operators by the factorization relation

$$W(t, p, q) * E(t, p, q) * W(0, 0, 0)^{-1} = \hat{W}(t, p, q) \in \exp \mathcal{G}_{\geq 0} \quad (5.2)$$

connecting $W = W(t, p, q)$ with the initial value at $(t, p, q) = (0, 0, 0)$. The generators of flows in $E(t, p, q)$ are noncommutative,

$$[\hbar^{-1} (\hbar \partial_x)^m y, \hbar^{-1} (\hbar \partial_x)^n z] = \hbar^{-1} (\hbar \partial_x)^{m+n}, \quad (5.3)$$

but note that commutators are all central. Thus the algebraic structure is very similar to Heisenberg algebras. This is a key to the following calculations.

Because of this mild noncommutativity, one can explicitly write down differential equations satisfied by $E(t, p, q)$ as:

$$\begin{aligned} \hbar \frac{\partial E(t, p, q)}{\partial t_n} &= (\hbar \partial_x)^n E(t, p, q), \\ \hbar \frac{\partial E(t, p, q)}{\partial p_n} &= \left(-z (\hbar \partial_x)^n - \frac{1}{2} \sum_{m=1}^{\infty} q_m (\hbar \partial_x)^{m+n} \right) * E(t, p, q), \\ \hbar \frac{\partial E(t, p, q)}{\partial q_n} &= \left(y (\hbar \partial_x)^n + \frac{1}{2} \sum_{m=1}^{\infty} p_m (\hbar \partial_x)^{m+n} \right) * E(t, p, q). \end{aligned} \quad (5.4)$$

[Proof: To derive these formulas, let us factorize $E(t, p, q)$ in two different ways:

$$\begin{aligned} E(t, p, q) &= \exp \hbar^{-1} \left(\sum_{n=1}^{\infty} t_n (\hbar \partial_x)^n - \sum_{n=1}^{\infty} p_n z (\hbar \partial_x)^n \right. \\ &\quad \left. - \frac{1}{2} \sum_{m,n=1}^{\infty} p_m q_n (\hbar \partial_x)^{m+n} \right) * \exp \hbar^{-1} \left(\sum_{n=1}^{\infty} q_n y (\hbar \partial_x)^n \right) \\ &= \exp \hbar^{-1} \left(\sum_{n=1}^{\infty} t_n (\hbar \partial_x)^n + \sum_{n=1}^{\infty} q_n y (\hbar \partial_x)^n \right. \\ &\quad \left. + \frac{1}{2} \sum_{m,n=1}^{\infty} p_m q_n (\hbar \partial_x)^{m+n} \right) * \exp \hbar^{-1} \left(- \sum_{n=1}^{\infty} p_n z (\hbar \partial_x)^n \right). \end{aligned} \quad (5.5)$$

Here we have used the Campbell–Hausdorff formula

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X, Y] + \dots\right) \tag{5.6}$$

for the star product. (Note that because of the above commutation relations of Heisenberg-type, multiple commutators disappear.) The first factorized form of $E(t, p, q)$ is suited for calculating p -derivatives, because p_n 's are included only in the first factor, and all terms included therein commute with each other. The second formula of (5.4) can thus be proven. Similarly, the second factorized form of $E(t, p, q)$ can be used to evaluate q -derivatives. For t -derivatives, nothing subtle is required. Q.E.D.]

Having these differential equations for $E(t, p, q)$, one can derive differential equations satisfied by the dressing operator as:

$$\begin{aligned} \hbar \frac{\partial W}{\partial t_n} * W^{-1} &= -\left(W * (\hbar \partial_x)^n * W^{-1}\right)_{\leq -1}, \\ \hbar \frac{\partial W}{\partial p_n} * W^{-1} &= -\left(-W * z (\hbar \partial_n)^n * W^{-1} \right. \\ &\quad \left. - \frac{1}{2} \sum_{m=1}^{\infty} q_m W * (\hbar \partial_x)^{m+n} * W^{-1}\right)_{\leq -1}, \\ \hbar \frac{\partial W}{\partial q_n} * W^{-1} &= -\left(W * y (\hbar \partial_x)^n * W^{-1} \right. \\ &\quad \left. + \frac{1}{2} \sum_{m=1}^{\infty} p_m W * (\hbar \partial_x)^{m+n} * W^{-1}\right)_{\leq -1}. \end{aligned} \tag{5.7}$$

[Proof: We simply repeat the reasoning in the end of Section 2. For instance, let us check the second equation. Differentiating the factorization relation and eliminating the initial value $W(0, 0, 0)$, one obtains the basic relation

$$\left(\frac{\partial W}{\partial p_n} + W * \frac{\partial E}{\partial p_n} * E^{-1}\right) * W^{-1} = \frac{\partial \hat{W}}{\partial p_n} * \hat{W}^{-1}. \tag{5.8}$$

Since the right hand side should be an element of $\mathcal{G}_{\geq 0}$,

$$\left(\text{left hand side of (5.7)}\right)_{\leq -1} = 0. \tag{5.9}$$

By use of (5.4), one can easily check that this gives the second equation of (5.7). The other equations can be proven in the same manner. Q.E.D.]

5.2. Lax and zero-curvature equations

We now define Lax operators just like the definition of M and V in the commuting hierarchy:

$$\begin{aligned}
L &= W * E(t, p, q) * \hbar \partial_x * E(t, p, q)^{-1} * W^{-1}, \\
M &= W * E(t, p, q) * x * E(t, p, q)^{-1} * W^{-1}, \\
U &= W * E(t, p, q) * y * E(t, p, q)^{-1} * W^{-1}, \\
V &= W * E(t, p, q) * z * E(t, p, q)^{-1} * W^{-1}.
\end{aligned} \tag{5.10}$$

“Pre-dressing” by $E(t, p, q)$ can be calculated by use of (4.11) as:

$$\begin{aligned}
E(t, p, q) * \hbar \partial_x * E(t, p, q)^{-1} &= \hbar \partial_x, \\
E(t, p, q) * x * E(t, p, q)^{-1} &= \sum_{n=1}^{\infty} n t_n (\hbar \partial_x)^{n-1} - \sum_{n=1}^{\infty} n p_n z (\hbar \partial_x)^{n-1} \\
&\quad + \sum_{n=1}^{\infty} n q_n y (\hbar \partial_x)^{n-1} - \frac{1}{2} \sum_{m, n=1}^{\infty} (m-n) p_m q_n (\hbar \partial_x)^{m+n-1} + x, \\
E(t, p, q) * y * E(t, p, q)^{-1} &= \sum_{n=1}^{\infty} p_n (\hbar \partial_x)^n + y, \\
E(t, p, q) * z * E(t, p, q)^{-1} &= \sum_{n=1}^{\infty} q_n (\hbar \partial_x)^n + z.
\end{aligned} \tag{5.11}$$

Dressing by W , then, yields the following expression of the Lax operators:

$$\begin{aligned}
L &= \hbar \partial_x + \sum_{n=1}^{\infty} g_{n+1} (\hbar \partial_x)^{-n}, \\
M &= \sum_{n=1}^{\infty} n t_n L^{n-1} - \sum_{n=1}^{\infty} n p_n V * L^{n-1} + \sum_{n=1}^{\infty} n q_n U * L^{n-1} \\
&\quad + \frac{1}{2} \sum_{m, n=1}^{\infty} (m-n) p_m q_n L^{m+n-1} + x + \sum_{n=1}^{\infty} h_{n+1} * L^{-n-1}, \\
U &= \sum_{n=1}^{\infty} p_n L^n + y + \sum_{n=1}^{\infty} u_n * L^{-n}, \\
V &= \sum_{n=1}^{\infty} q_n L^n + z + \sum_{n=1}^{\infty} v_n * L^{-n}.
\end{aligned} \tag{5.12}$$

Just as in the commuting hierarchy, cf. (4.10) and (4.21), the coefficients on the right hand sides have a smooth limit as $\hbar \rightarrow 0$.

Let us now write down differential equations satisfied by these Lax operators. First note that (5.7) can now be written

$$\begin{aligned}
 \hbar \frac{\partial W}{\partial t_n} &= -(L^n)_{\leq -1} * W, \\
 \hbar \frac{\partial W}{\partial p_n} &= -\left(-V * L^n + \frac{1}{2} \sum_{m=1}^{\infty} q_m L^{m+n}\right)_{\leq -1} * W, \\
 \hbar \frac{\partial W}{\partial q_n} &= -\left(U * L^n - \frac{1}{2} \sum_{m=1}^{\infty} p_m L^{m+n}\right)_{\leq -1} * W.
 \end{aligned}
 \tag{5.13}$$

Remarkably, M does not occur on the right hand side. This implies that one does not actually need M for defining Zakharov–Shabat operators. One can indeed derive, from the above equations, the Lax equations

$$\begin{aligned}
 \hbar \frac{\partial L}{\partial t_n} &= [B_n, L], & \hbar \frac{\partial L}{\partial p_n} &= [C_n, L], & \hbar \frac{\partial L}{\partial q_n} &= [D_n, L], \\
 \hbar \frac{\partial U}{\partial t_n} &= [B_n, U], & \hbar \frac{\partial U}{\partial p_n} &= [C_n, U], & \hbar \frac{\partial U}{\partial q_n} &= [D_n, U], \\
 \hbar \frac{\partial V}{\partial t_n} &= [B_n, V], & \hbar \frac{\partial V}{\partial p_n} &= [C_n, V], & \hbar \frac{\partial V}{\partial q_n} &= [D_n, V].
 \end{aligned}
 \tag{5.14}$$

The Zakharov–Shabat operators B_n , C_n and D_n are given by

$$\begin{aligned}
 B_n &= (L^n)_{\geq 0}, \\
 C_n &= \left(-V * L^n + \frac{1}{2} \sum_{m=1}^{\infty} q_m L^{m+n}\right)_{\geq 0}, \\
 D_n &= \left(U * L^n - \frac{1}{2} \sum_{m=1}^{\infty} p_m L^{m+n}\right)_{\geq 0},
 \end{aligned}
 \tag{5.15}$$

and satisfy the zero-curvature equations

$$\begin{aligned}
 \hbar \frac{\partial B_m}{\partial t_n} - \hbar \frac{\partial B_n}{\partial t_m} + [B_m, B_n] &= 0, \\
 \hbar \frac{\partial B_m}{\partial p_n} - \hbar \frac{\partial C_n}{\partial t_m} + [B_m, C_n] &= 0, \\
 \hbar \frac{\partial B_m}{\partial q_n} - \hbar \frac{\partial D_n}{\partial t_m} + [B_m, D_n] &= 0, \\
 \hbar \frac{\partial C_m}{\partial p_n} - \hbar \frac{\partial C_n}{\partial p_m} + [C_m, C_n] &= 0, \\
 \hbar \frac{\partial C_m}{\partial q_n} - \hbar \frac{\partial D_n}{\partial p_m} + [C_m, D_n] &= 0, \\
 \hbar \frac{\partial D_m}{\partial q_n} - \hbar \frac{\partial D_n}{\partial q_m} + [D_m, D_n] &= 0.
 \end{aligned}
 \tag{5.16}$$

Thus the Lax and zero-curvature equations are closed within L , U and V . (Of course M , too, satisfies the same Lax equations as the other three Lax oper-

ators.) Let us focus the following consideration on these three Lax operators. Note that they obey the commutation relations

$$[L, U] = [L, V] = 0, \quad [U, V] = \hbar. \quad (5.17)$$

6. Reduction to selfdual gravity

Moyal algebraic deformations of selfdual gravity emerge from the hierarchy of (t, p, q) flows as a kind of “dimensional reduction”. Actually, connection to selfdual gravity takes two distinct forms, which correspond to two different local expressions of selfdual gravity—the first and second heavenly equations of Plebanski [13].

6.1. Constraints and reduced variables

The reduction of the (t, p, q) flows is defined by the constraints

$$\frac{\partial w_n}{\partial x} = 0, \quad n = 1, 2, \dots \quad (6.1)$$

on the coefficients of the dressing operator. This forces L and B_n to be trivial,

$$L = \hbar \partial_x, \quad B_n = (\hbar \partial_x)^n. \quad (6.2)$$

In particular,

$$\frac{\partial W}{\partial t_n} = 0, \quad \frac{\partial L}{\partial t_n} = \frac{\partial U}{\partial t_n} = \frac{\partial V}{\partial t_n} = 0. \quad (6.3)$$

Under these constraints, $\hbar \partial_x$ may be replaced by a “spectral parameter” λ . The dressing operator then turns into a Laurent series with Moyal algebraic coefficients:

$$W(\lambda) = 1 + \sum_{n=1}^{\infty} w_n(\hbar, t, p, q, y, z) \lambda^{-n}. \quad (6.4)$$

This type of constraint is also used in the theory of multi-component KP hierarchies [8] to derive an N -component version of the AKNS (Ablowitz–Kaup–Newell–Segur) or ZS (Zakharov–Shabat) hierarchy. The role of dressing operators is played therein by a Laurent series with matrix coefficients:

$$W(\lambda) = 1 + \sum_{n=1}^{\infty} w_n \lambda^{-n}, \quad w_n(t) \in \mathfrak{gl}(N). \quad (6.5)$$

The Lax and Zakharov–Shabat operators, too, can be represented by Laurent series. As noted above, L is now trivial. Nontrivial dynamical contents are carried by U and V , which are now replaced by Laurent series of the form

$$\begin{aligned}
 U(\lambda) &= \sum_{n=1}^{\infty} p_n \lambda^n + y + \sum_{n=1}^{\infty} u_n \lambda^{-n}, \\
 V(\lambda) &= \sum_{n=1}^{\infty} q_n \lambda^n + z + \sum_{n=1}^{\infty} v_n \lambda^{-n}.
 \end{aligned}
 \tag{6.6}$$

Algebraic relations among the Lax operators are carried over to these Laurent series. For instance, $U(\lambda)$ and $V(\lambda)$ obey the canonical commutation relation

$$[U(\lambda), V(\lambda)] = \hbar
 \tag{6.7}$$

and the dressing relations

$$\begin{aligned}
 U(\lambda) &= W(\lambda) * \left(\sum_{n=1}^{\infty} p_n \lambda^n + y \right) * W(\lambda)^{-1} \\
 &= \sum_{n=1}^{\infty} p_n \lambda^n + W(\lambda) * y * W(\lambda)^{-1}, \\
 V(\lambda) &= W(\lambda) * \left(\sum_{n=1}^{\infty} q_n \lambda^n + z \right) * W(\lambda)^{-1} \\
 &= \sum_{n=1}^{\infty} q_n \lambda^n + W(\lambda) * z * W(\lambda)^{-1}
 \end{aligned}
 \tag{6.8}$$

Laurent series corresponding to Zakharov–Shabat operators will be given in the next subsection.

6.2. Equations of flows in terms of Laurent series

The factorization relation connecting W and its initial value is now translated into a factorization relation for $W(\lambda) = W(p, q, \lambda)$ and its initial value at $p = q = 0$:

$$W(p, q, \lambda) * E(p, q, \lambda) * W(0, 0, \lambda)^{-1} = \hat{W}(p, q, \lambda),
 \tag{6.9}$$

where $\hat{W}(p, q, \lambda)$ is a Laurent series to be obtained from \hat{W} by replacing $\hbar \partial_x \rightarrow \lambda$ (which, too, turns out to be independent of x and t), and $E(p, q, \lambda)$ is the Laurent series

$$E(p, q, \lambda) = \exp_* \hbar^{-1} \left(- \sum_{n=1}^{\infty} p_n z \lambda^n + \sum_{n=1}^{\infty} q_n y \lambda^n \right).
 \tag{6.10}$$

(We put $t = 0$ here and in the following.) The Lie algebra \mathcal{G} of pseudo-differential operators should thus be redefined to be a Lie algebra of Laurent series with Moyal algebraic coefficients:

$$\begin{aligned} \mathcal{G} &= \{A \mid A = \sum a_n(\hbar, y, z)\lambda^n\} = \mathcal{G}_{\geq 0} \oplus \mathcal{G}_{\leq -1}, \\ (\)_+ &= (\)_{\geq 0} : \text{projection onto } \lambda^0, \lambda^1, \dots, \\ (\)_- &= (\)_{\leq -1} : \text{projection onto } \lambda^{-1}, \lambda^{-2}, \dots. \end{aligned} \quad (6.11)$$

Differential equations for the Lax and dressing operators, too, can be translated into the language of Laurent series. First, $W(\lambda)$ satisfy the equations

$$\begin{aligned} \hbar \frac{\partial W(\lambda)}{\partial p_n} &= (V(\lambda)\lambda^n)_{\leq -1} * W(\lambda), \\ \hbar \frac{\partial W(\lambda)}{\partial q_n} &= -(U(\lambda)\lambda^n)_{\leq -1} * W(\lambda). \end{aligned} \quad (6.12)$$

Similarly, $U(\lambda)$ and $V(\lambda)$ satisfy the Lax equations

$$\begin{aligned} \hbar \frac{\partial U(\lambda)}{\partial p_n} &= [C_n(\lambda), U(\lambda)] = [-(V(\lambda)\lambda^n)_{\geq 0}, U(\lambda)], \\ \hbar \frac{\partial U(\lambda)}{\partial q_n} &= [D_n(\lambda), U(\lambda)] = [(U(\lambda)\lambda^n)_{\geq 0}, U(\lambda)], \\ \hbar \frac{\partial V(\lambda)}{\partial p_n} &= [C_n(\lambda), V(\lambda)] = [-(V(\lambda)\lambda^n)_{\geq 0}, V(\lambda)], \\ \hbar \frac{\partial V(\lambda)}{\partial q_n} &= [D_n(\lambda), V(\lambda)] = [(U(\lambda)\lambda^n)_{\geq 0}, V(\lambda)], \end{aligned} \quad (6.13)$$

where $C_n(\lambda)$ and $D_n(\lambda)$ are given by

$$\begin{aligned} C_n(\lambda) &= -(V(\lambda)\lambda^n)_{\geq 0} + \frac{1}{2} \sum_{m=1}^{\infty} q_m \lambda^{m+n}, \\ D_n(\lambda) &= (U(\lambda)\lambda^n)_{\geq 0} - \frac{1}{2} \sum_{m=1}^{\infty} p_m \lambda^{m+n} \end{aligned} \quad (6.14)$$

and obey the zero-curvature equations

$$\begin{aligned} \hbar \frac{\partial C_m(\lambda)}{\partial p_n} - \hbar \frac{\partial C_n(\lambda)}{\partial p_m} + [C_m(\lambda), C_n(\lambda)] &= 0, \\ \hbar \frac{\partial C_m(\lambda)}{\partial q_n} - \hbar \frac{\partial D_n(\lambda)}{\partial p_m} + [C_m(\lambda), D_n(\lambda)] &= 0, \\ \hbar \frac{\partial D_m(\lambda)}{\partial q_n} - \hbar \frac{\partial D_n(\lambda)}{\partial q_m} + [D_m(\lambda), D_n(\lambda)] &= 0. \end{aligned} \quad (6.15)$$

6.3. Relation to the second heavenly equation

We now show that the above reduced hierarchy is related to a Moyal algebraic deformation of selfdual gravity. This deformation corresponds to Plebanski's "second heavenly equation" [13]:

$$\frac{\partial^2 \Theta}{\partial y \partial q} - \frac{\partial^2 \Theta}{\partial z \partial p} + \left\{ \frac{\partial \Theta}{\partial y}, \frac{\partial \Theta}{\partial z} \right\} = 0. \tag{6.16}$$

Strachan’s equation [6], on the other hand, is a deformation of Plebanski’s “first heavenly equation” [13]:

$$\left\{ \frac{\partial \Omega}{\partial p}, \frac{\partial \Omega}{\partial q} \right\}^{\wedge} = \frac{\partial^2 \Omega}{\partial p \partial \hat{p}} \frac{\partial^2 \Omega}{\partial q \partial \hat{q}} - \frac{\partial^2 \Omega}{\partial p \partial \hat{q}} \frac{\partial^2 \Omega}{\partial q \partial \hat{p}} = 1, \tag{6.17}$$

where (p, q, \hat{p}, \hat{q}) are another set of variables, and $\{ \ , \ }^{\wedge}$ stands for the Poisson bracket in (\hat{p}, \hat{q}) . Comparing the above hierarchy with hierarchies constructed in Ref. [10], and identifying

$$p_1 = p, \quad q_1 = q, \tag{6.18}$$

one will see that the above hierarchy gives a Moyal algebraic deformation of a hierarchy constructed therein for the second heavenly equation. Furthermore, Plebanski’s Θ potential turns out to be linked with the next-to-leading term w_1 of $W(\lambda)$:

$$\Theta = -\hbar w_1. \tag{6.19}$$

Let us show how to deduce these facts. To this end, we rewrite (6.12) as

$$\begin{aligned} \hbar \frac{\partial W(\lambda)}{\partial p_n} * W(\lambda)^{-1} &= (V(\lambda) \lambda^n)_{\leq -1}, \\ \hbar \frac{\partial W(\lambda)}{\partial q_n} * W(\lambda)^{-1} &= -(U(\lambda) \lambda^n)_{\leq -1}, \end{aligned} \tag{6.20}$$

and extract λ^{-1} terms of both sides. This results in the relations

$$\hbar \frac{\partial w_1}{\partial p_n} = v_{n+1}, \quad \hbar \frac{\partial w_1}{\partial q_n} = -u_{n+1}. \tag{6.21}$$

[Actually, these relations can be extended to the case of $n = 0$ by identifying

$$p_0 = y, \quad q_0 = z, \tag{6.22}$$

as one can directly check from the definition of $U(\lambda)$ and $V(\lambda)$.] In terms of the function Θ defined by (6.19), these relations can be rewritten

$$u_n = \frac{\partial \Theta}{\partial q_{n-1}}, \quad v_n = -\frac{\partial \Theta}{\partial p_{n-1}}. \tag{6.23}$$

In fact, exactly the same relations for selfdual gravity have been derived in Ref. [10]. Furthermore, repeating the technical calculations presented therein, one can show that Eqs. (6.15) are actually equivalent to:

$$\begin{aligned}
\frac{\partial v_m}{\partial p_n} - \frac{\partial v_n}{\partial p_m} + \{v_n, v_m\}_{\hbar} &= 0, \\
\frac{\partial u_m}{\partial p_n} + \frac{\partial v_n}{\partial q_m} + \{v_n, u_m\}_{\hbar} &= 0, \\
\frac{\partial u_m}{\partial p_n} - \frac{\partial u_n}{\partial p_m} - \{u_n, u_m\}_{\hbar} &= 0, \quad m, n = 0, 1, \dots
\end{aligned} \tag{6.24}$$

By (6.23), these equations can be rewritten

$$\begin{aligned}
\frac{\partial^2 \Theta}{\partial p_n \partial q_{m-1}} - \frac{\partial^2 \Theta}{\partial p_m \partial q_{n-1}} - \left\{ \frac{\partial \Theta}{\partial p_{n-1}}, \frac{\partial \Theta}{\partial p_{m-1}} \right\}_{\hbar} &= 0, \\
\frac{\partial^2 \Theta}{\partial p_n \partial q_{m-1}} - \frac{\partial^2 \Theta}{\partial q_m \partial p_{n-1}} - \left\{ \frac{\partial \Theta}{\partial p_{n-1}}, \frac{\partial \Theta}{\partial q_{m-1}} \right\}_{\hbar} &= 0, \\
\frac{\partial^2 \Theta}{\partial q_n \partial q_{m-1}} - \frac{\partial^2 \Theta}{\partial q_m \partial q_{n-1}} - \left\{ \frac{\partial \Theta}{\partial q_{n-1}}, \frac{\partial \Theta}{\partial q_{m-1}} \right\}_{\hbar} &= 0.
\end{aligned} \tag{6.25}$$

The second equation with $m = n = 1$ gives a Moyal algebraic deformation of the second heavenly equation (6.16). In the limit of $\hbar \rightarrow 0$, the Moyal bracket $\{ \ , \ }_{\hbar}$ turns into the Poisson bracket $\{ \ , \ }$, and these equations reproduce a hierarchy constructed in Ref. [10].

6.4. Relation to the first heavenly equation

The relation to the first heavenly equation is more involved. The first heavenly equation (6.17) is written in terms of the Poisson bracket $\{ \ , \ }_{\hbar}$ in (\hat{p}, \hat{q}) , and Strachan's idea is to deform this bracket into a Moyal bracket as:

$$\left\{ \frac{\partial \Omega}{\partial p}, \frac{\partial \Omega}{\partial q} \right\}_{\hbar}^{\wedge} = \frac{\partial^2 \Omega}{\partial p \partial \hat{p}} \frac{\partial^2 \Omega}{\partial q \partial \hat{q}} - \frac{\partial^2 \Omega}{\partial p \partial \hat{q}} \frac{\partial^2 \Omega}{\partial q \partial \hat{p}} + O(\hbar) = 1. \tag{6.26}$$

These brackets differ from our brackets in (y, z) . Nevertheless, very curiously and remarkably, it turns out that a similar Moyal algebraic analogue of the first heavenly equation emerges in a “gauge transformation” of the present hierarchy.

Let us first specify this gauge transformation. By “gauge transformations” we mean symmetries of Eqs. (6.12)–(6.15) given by

$$\begin{aligned}
W(\lambda) &\rightarrow W^g(\lambda) = g^{-1} * W(\lambda), \\
U(\lambda) &\rightarrow U^g(\lambda) = g^{-1} * U(\lambda) * g, \\
V(\lambda) &\rightarrow V^g(\lambda) = g^{-1} * V(\lambda) * g,
\end{aligned}$$

$$\begin{aligned}
 C_n(\lambda) &\rightarrow C_n^g(\lambda) = g^{-1} * C_n(\lambda) * g - g^{-1} * \frac{\partial g}{\partial p_n}, \\
 D_n(\lambda) &\rightarrow D_n^g(\lambda) = g^{-1} * D_n(\lambda) * g - g^{-1} * \frac{\partial g}{\partial q_n},
 \end{aligned}
 \tag{6.27}$$

where g is a function of (y, z, p, q) and independent of λ . With suitable choice of g (i.e., “a gauge-fixing”), λ^0 terms in $C_n^g(\lambda)$ and $D_n(\lambda)$ can be eliminated. Such a gauge-fixing is realized by

$$g = \hat{w}_0, \tag{6.28}$$

where \hat{w}_0 is the leading term of the Laurent series $\hat{W}(\lambda)$ that arises in the factorization relation:

$$\hat{W}(\lambda) = \hat{w}_0 + \hat{w}_1 \lambda^1 + \dots \tag{6.29}$$

The factorization relation, too, is gauge-covariant if $\hat{W}(\lambda)$ is also transformed as

$$\hat{W}(\lambda) \rightarrow \hat{W}^g(\lambda) = g^{-1} * \hat{W}(\lambda). \tag{6.30}$$

In the language of the factorization relation, the above special gauge transformation simply means changing a normalization condition of solutions. By the above gauge transformation (note that $g|_{p=q=0} = 1$), indeed, the original normalization condition

$$W(\lambda)|_{\lambda=\infty} = 1 \tag{6.31}$$

is converted into a new condition of the form

$$\hat{W}^g(\lambda)|_{\lambda=0} = 1. \tag{6.32}$$

This corresponds to taking a new direct sum decomposition of \mathcal{G} :

$$\begin{aligned}
 \mathcal{G} &= \mathcal{G}_{\geq 1} \oplus \mathcal{G}_{\leq 0}, \\
 (\)_+ &= (\)_{\geq 0} : \text{projection onto } \lambda^1, \lambda^2, \dots, \\
 (\)_- &= (\)_{\leq 0} : \text{projection onto } \lambda^0, \lambda^{-1}, \dots
 \end{aligned}
 \tag{6.33}$$

Repeating previous calculations in this new setting, one can derive the following equations, which are gauge transformations of (6.12):

$$\begin{aligned}
 \hbar \frac{\partial W^g(\lambda)}{\partial p_n} &= \left(V^g(\lambda) \lambda^n \right)_{\leq 0} * W^g(\lambda), \\
 \hbar \frac{\partial W^g(\lambda)}{\partial q_n} &= - \left(U^g(\lambda) \lambda^n \right)_{\leq 0} * W^g(\lambda).
 \end{aligned}
 \tag{6.34}$$

This kind of gauge transformation is also known for the KP hierarchy etc. [14].

Let us now return to the issue of selfdual gravity. We now argue that the function

$$\Omega = -\hbar \hat{w}_1^g \tag{6.35}$$

satisfies a Moyal algebraic analogue of the first heavenly equation,

$$\left\{ \frac{\partial \Omega}{\partial p}, \frac{\partial \Omega}{\partial q} \right\}_{\hbar} = \frac{\partial^2 \Omega}{\partial p \partial y} \frac{\partial^2 \Omega}{\partial q \partial z} - \frac{\partial^2 \Omega}{\partial p \partial z} \frac{\partial^2 \Omega}{\partial q \partial y} + O(\hbar) = 1, \tag{6.36}$$

where (p_1, q_1) are identified with (p, q) as in (6.18), and \hat{w}_1^g is the next-to-leading term of Laurent expansion of $\hat{W}^g(\lambda)$,

$$\hat{W}^g(\lambda) = 1 + \hat{w}_1^g \lambda + \dots \tag{6.37}$$

Let us show details. As a consequence of the factorization relation, $\hat{W}^g(\lambda)$ satisfies the same equation as $W^g(\lambda)$:

$$\begin{aligned} \hbar \frac{\partial \hat{W}^g(\lambda)}{\partial p_n} &= \left(V^g(\lambda) \lambda^n \right)_{\leq 0} * \hat{W}^g(\lambda), \\ \hbar \frac{\partial \hat{W}^g(\lambda)}{\partial q_n} &= - \left(U^g(\lambda) \lambda^n \right)_{\leq 0} * \hat{W}^g(\lambda). \end{aligned} \tag{6.38}$$

Then imitating the derivation of (6.23), we can show that

$$u_n^g = - \frac{\partial \Omega}{\partial q_{n+1}}, \quad v_n^g = \frac{\partial \Omega}{\partial p_{n+1}}, \tag{6.39}$$

where u_n^g and v_n^g are Laurent coefficients of $U^g(\lambda)$ and $V^g(\lambda)$,

$$\begin{aligned} U^g(\lambda) &= \sum_{n=1}^{\infty} p_n \lambda^n + u_0^g + \sum_{n=1}^{\infty} u_n^g \lambda^{-n}, \\ V^g(\lambda) &= \sum_{n=1}^{\infty} q_n \lambda^n + v_0^g + \sum_{n=1}^{\infty} v_n^g \lambda^{-n}. \end{aligned} \tag{6.40}$$

In particular,

$$u_0^g = - \frac{\partial \Omega}{\partial q_1}, \quad v_0^g = \frac{\partial \Omega}{\partial p_1}. \tag{6.41}$$

On the other hand, by the construction, u_0^g and v_0^g are linked with the canonically conjugate pair y and z as

$$u_0^g = g^{-1} * y * g, \quad v_0^g = g^{-1} * z * g, \tag{6.42}$$

hence canonically conjugate in themselves:

$$\{u_0^g, v_0^g\}_{\hbar} = 1. \tag{6.43}$$

Substitution of (6.41) into this relation yields (6.36). Furthermore, as in the case of Θ , one can derive an infinite system of equations for Ω . Those

equations give a Moyal algebraic deformation of another hierarchy constructed in Ref. [10].

We are thus in a somewhat puzzled situation. Our Moyal algebraic deformation (6.36) has exactly the same structure as Strachan's equation (6.26), but the Moyal brackets are living in apparently different planes, i.e., (y, z) and (\hat{p}, \hat{q}) . In the standard treatment of selfdual gravity, the two coordinate systems (p, q, y, z) and (p, q, \hat{p}, \hat{q}) are distinct and never identical (unless the spacetime is flat).

This problem will be resolved in a Toda version of our Moyal algebraic nonabelian KP hierarchies. The dressing operator approach of Ref. [7], actually employs two dressing operators rather than a single one as we now considering. This situation resembles the Toda hierarchy [15], which, too, is based on two dressing operators. (Actually, although not clearly mentioned therein, the fundamental ideas in Ref. [7] are rather borrowed from the theory of the Toda hierarchy.) A natural framework in which to embed Strachan's equation is thus a Moyal algebraic version of the Toda hierarchy.

7. Quasi-classical limit

The Moyal algebraic hierarchies of both commuting and noncommuting types turn out to have a quasi-classical ($\hbar \rightarrow 0$) limit. The hierarchies in this limit, too, allow Lax- and zero-curvature-type representations, but commutators in the ordinary framework are now replaced by Poisson brackets. The situation is parallel to the KP hierarchy; its quasi-classical limit is the dispersionless KP hierarchy, and Poisson brackets are two dimensional. In the present setting, Poisson brackets are four dimensional.

7.1. Prescription of the quasi-classical limit

Let us recall the derivation of the dispersionless KP hierarchy as the quasi-classical limit of the ordinary KP hierarchy. The canonically conjugate Lax operators L and M of the KP hierarchy [12], in the presence of \hbar , can be written

$$L = \hbar \partial_x + \sum_{n=1}^{\infty} g_{n+1} (\hbar \partial_x)^{-n},$$

$$M = \sum_{n=1}^{\infty} n t_n L^{n-1} + x + \sum_{n=1}^{\infty} h_{n+1} L^{-n-1}, \quad (7.1)$$

and satisfy the canonical commutation relation

$$[L, M] = \hbar \quad (7.2)$$

and the Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \frac{\partial M}{\partial t_n} = [B_n, M], \tag{7.3}$$

where

$$B_n = (L^n)_{\geq 0}. \tag{7.4}$$

The coefficients are assumed to behave smoothly as $\hbar \rightarrow 0$:

$$g_n = g_n^{(0)}(t, x) + O(\hbar), \quad h_n = h_n^{(0)}(t, x) + O(\hbar). \tag{7.5}$$

In the limit of $\hbar \rightarrow 0$, the canonically conjugate pair $(\hbar \partial_x, x)$ of operators is replaced by a canonically conjugate pair (k, x) of coordinates on a two dimensional symplectic manifold. The role of the Lax operators L and M is then played by the Laurent series

$$\begin{aligned} \mathcal{L} &= k + \sum_{n=1}^{\infty} g_{n+1}^{(0)} k^{-n}, \\ \mathcal{M} &= \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{n=1}^{\infty} h_{n+1}^{(0)} \mathcal{L}^{-n-1}, \end{aligned} \tag{7.6}$$

which satisfy the canonical Poisson relation

$$\{\mathcal{L}, \mathcal{M}\} = 1 \tag{7.7}$$

and the Lax-like equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{B_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{B_n, \mathcal{M}\} \tag{7.8}$$

with respect to the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial k} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial k}. \tag{7.9}$$

Here the B_n 's are polynomials of the form

$$B_n = (\mathcal{L}^n)_{\geq 0}, \tag{7.10}$$

and the projection operators $(\)_{\geq 0}$ and $(\)_{\leq -1}$ are defined by

$$\begin{aligned} (\)_{\geq 0} &: \text{projection onto } k^0, k^1, \dots, \\ (\)_{\leq -1} &: \text{projection onto } k^{-1}, k^{-2}, \dots. \end{aligned} \tag{7.11}$$

These polynomials B_n then satisfy the zero-curvature-like equations

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + \{B_m, B_n\} = 0. \tag{7.12}$$

Thus one can reproduce the Lax formalism of the dispersionless KP hierarchy.

In the case of our higher dimensional hierarchies, we have another canonically conjugate pair (y, z) that commute with $(\hbar\partial_x, x)$, hence altogether four operators $(\hbar\partial_x, x, y, z)$. These operators obey the canonical commutation relations

$$[\hbar\partial_x, x] = \hbar, \quad [y, z] = \hbar, \\ \text{other commutators} = 0. \tag{7.13}$$

In the limit of $\hbar \rightarrow 0$, these operators will be replaced by canonical coordinates (k, x, y, z) on a four dimensional symplectic manifold, and by the standard rule

$$\hbar^{-1} [\quad , \quad] \longrightarrow \{ \{ \quad , \quad \} \} \tag{7.14}$$

of quantum mechanics, the commutator of operators will be reduced to the four dimensional Poisson bracket

$$\{ \{ A, B \} \} = \frac{\partial A}{\partial k} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial k} + \frac{\partial A}{\partial y} \frac{\partial B}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial B}{\partial y} \tag{7.15}$$

of functions of (k, x, y, z) . The role of Lax and Zakharov–Shabat operators will thus be played by functions of (k, x, y, z) . Those classical analogues of Lax and Zakharov–Shabat operators will satisfy Poisson algebraic analogues of the Lax and zero-curvature equations with respect to $\{ \{ \quad , \quad \} \}$. Let us show details below.

7.2. Hierarchy of commuting flows

For the hierarchy of $t_{n\alpha}$ flows, classical counterparts of L, M, U and V are given by the Laurent series

$$\mathcal{L} = k + \sum_{n=1}^{\infty} g_{n+1}^{(0)} k^{-n}, \\ \mathcal{M} = \sum_{n,\alpha} n t_{n\alpha} \mathcal{L}^{n-1} \mathcal{U}^\alpha + x + \sum_{n=1}^{\infty} h_{n+1}^{(0)} \mathcal{L}^{-n-1}, \\ \mathcal{U} = y + \sum_{n=1}^{\infty} u_n^{(0)} \mathcal{L}^{-n}, \\ \mathcal{V} = \sum_{n,\alpha} \alpha t_{n\alpha} \mathcal{L}^n \mathcal{U}^{\alpha-1} + z + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \tag{7.16}$$

where the coefficients $g_n^{(0)}, h_n^{(0)}, u_n^{(0)}$ and $v_n^{(0)}$ are the leading terms of the \hbar expansion in (4.10) are (4.21). These Laurent series satisfy the canonical Poisson relations

$$\begin{aligned} \{\{\mathcal{L}, \mathcal{M}\}\} &= \{\{\mathcal{U}, \mathcal{V}\}\} = 1, \\ \text{other Poisson brackets} &= 0, \end{aligned} \tag{7.17}$$

and the Poisson algebraic Lax equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_{n\alpha}} &= \{\{\mathcal{B}_{n\alpha}, \mathcal{L}\}\}, & \frac{\partial \mathcal{M}}{\partial t_{n\alpha}} &= \{\{\mathcal{B}_{n\alpha}, \mathcal{M}\}\}, \\ \frac{\partial \mathcal{U}}{\partial t_{n\alpha}} &= \{\{\mathcal{B}_{n\alpha}, \mathcal{U}\}\}, & \frac{\partial \mathcal{V}}{\partial t_{n\alpha}} &= \{\{\mathcal{B}_{n\alpha}, \mathcal{V}\}\}. \end{aligned} \tag{7.18}$$

Here the classical counterparts $\mathcal{B}_{n\alpha}$ of the Zakharov–Shabat operators $B_{n\alpha}$ are now given by

$$\mathcal{B}_{n\alpha} = (\mathcal{L}^n \mathcal{U}^\alpha)_{\geq 0}, \tag{7.19}$$

where

$$\begin{aligned} ()_{\geq 0} &: \text{projection onto } k^0, k^1, \dots, \\ ()_{\leq -1} &: \text{projection onto } k^{-1}, k^{-2}, \dots, \end{aligned} \tag{7.20}$$

and satisfy the Poisson algebraic zero-curvature equations

$$\frac{\partial \mathcal{B}_{m\alpha}}{\partial t_{n\beta}} - \frac{\partial \mathcal{B}_{n\beta}}{\partial t_{m\alpha}} + \{\{\mathcal{B}_{m\alpha}, \mathcal{B}_{n\beta}\}\} = 0. \tag{7.21}$$

Note that if we now impose the constraints

$$\begin{aligned} u_n^{(0)} &= 0, \quad v_n^{(0)} = 0 \text{ for } n = 1, 2, \dots \\ (\Leftrightarrow \mathcal{L} = y, \mathcal{V} = \sum \alpha t_{n\alpha} \mathcal{L}^n y^{\alpha-1} + z) \end{aligned} \tag{7.22}$$

only the t_{n0} flows remain nontrivial. The reduced hierarchy is substantially the same as the dispersionless KP hierarchy.

7.3. Hierarchy of noncommuting flows

For the hierarchy of (t, p, q) flows, classical counterparts of L, U and V are given by

$$\begin{aligned} \mathcal{L} &= k + \sum_{n=1}^{\infty} g_{n+1}^{(0)} k^{-n}, \\ \mathcal{U} &= \sum_{n=1}^{\infty} p_n \mathcal{L}^n + y + \sum_{n=1}^{\infty} u_n^{(0)} \mathcal{L}^{-n}, \\ \mathcal{V} &= \sum_{n=1}^{\infty} q_n \mathcal{L}^n + z + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}. \end{aligned} \tag{7.23}$$

They satisfy the Poisson bracket relations

$$\{\{\mathcal{L}, \mathcal{U}\}\} = \{\{\mathcal{L}, \mathcal{V}\}\} = 0, \quad \{\{\mathcal{U}, \mathcal{V}\}\} = 1, \tag{7.24}$$

and the Poisson algebraic Lax equations

$$\begin{aligned} \hbar \frac{\partial \mathcal{L}}{\partial t_n} &= \{\{\mathcal{B}_n, \mathcal{L}\}\}, & \hbar \frac{\partial \mathcal{L}}{\partial p_n} &= \{\{\mathcal{C}_n, \mathcal{L}\}\}, & \hbar \frac{\partial \mathcal{L}}{\partial q_n} &= \{\{\mathcal{D}_n, \mathcal{L}\}\}, \\ \hbar \frac{\partial \mathcal{U}}{\partial t_n} &= \{\{\mathcal{B}_n, \mathcal{U}\}\}, & \hbar \frac{\partial \mathcal{U}}{\partial p_n} &= \{\{\mathcal{C}_n, \mathcal{U}\}\}, & \hbar \frac{\partial \mathcal{U}}{\partial q_n} &= \{\{\mathcal{D}_n, \mathcal{U}\}\}, \\ \hbar \frac{\partial \mathcal{V}}{\partial t_n} &= \{\{\mathcal{B}_n, \mathcal{V}\}\}, & \hbar \frac{\partial \mathcal{V}}{\partial p_n} &= \{\{\mathcal{C}_n, \mathcal{V}\}\}, & \hbar \frac{\partial \mathcal{V}}{\partial q_n} &= \{\{\mathcal{D}_n, \mathcal{V}\}\}, \end{aligned} \tag{7.25}$$

where \mathcal{B}_n , \mathcal{C}_n and \mathcal{D} are given by

$$\begin{aligned} \mathcal{B}_n &= (\mathcal{L}^n)_{\geq 0}, \\ \mathcal{C}_n &= \left(-\mathcal{V}\mathcal{L}^n + \frac{1}{2} \sum_{m=1}^{\infty} q_m \mathcal{L}^{m+n}\right)_{\geq 0}, \\ \mathcal{D}_n &= \left(\mathcal{U}\mathcal{L}^n - \frac{1}{2} \sum_{m=1}^{\infty} p_m \mathcal{L}^{m+n}\right)_{\geq 0}, \end{aligned} \tag{7.26}$$

and satisfy the Poisson algebraic zero-curvature equations

$$\begin{aligned} \frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\{\mathcal{B}_m, \mathcal{B}_n\}\} &= 0, \\ \frac{\partial \mathcal{B}_m}{\partial p_n} - \frac{\partial \mathcal{C}_n}{\partial t_m} + \{\{\mathcal{B}_m, \mathcal{C}_n\}\} &= 0, \\ \frac{\partial \mathcal{B}_m}{\partial q_n} - \frac{\partial \mathcal{D}_n}{\partial t_m} + \{\{\mathcal{B}_m, \mathcal{D}_n\}\} &= 0, \\ \frac{\partial \mathcal{C}_m}{\partial p_n} - \frac{\partial \mathcal{C}_n}{\partial p_m} + \{\{\mathcal{C}_m, \mathcal{C}_n\}\} &= 0, \\ \frac{\partial \mathcal{C}_m}{\partial q_n} - \frac{\partial \mathcal{D}_n}{\partial p_m} + \{\{\mathcal{C}_m, \mathcal{D}_n\}\} &= 0, \\ \frac{\partial \mathcal{D}_m}{\partial q_n} - \frac{\partial \mathcal{D}_n}{\partial q_m} + \{\{\mathcal{D}_m, \mathcal{D}_n\}\} &= 0. \end{aligned} \tag{7.27}$$

Selfdual gravity can now be reproduced under the constraints

$$g_{n+1}^{(0)} = 1 \quad \text{for } n = 1, 2, \dots \quad \left(\iff \mathcal{L} = k\right). \tag{7.28}$$

All t_n flows then become trivial, and upon identifying $\lambda = k$, the reduced hierarchy of remaining (p, q) flows reproduces the same hierarchy as constructed in Ref. [10] for the second heavenly equation. Note that in the previous section, we first imposed constraints, then took the $\hbar \rightarrow 0$ limit. In this section,

we first took the $\hbar \rightarrow 0$ limit, then put the constraints. We have thus two routes to the same second heavenly equation.

The status of the first heavenly equation is rather obscure from this point view. As mentioned in the last section, a better approach will be to start from a Toda version of the hierarchy of (t, p, q) flows, and to consider quasi-classical limit. Similarly, a Toda version of the hierarchy of t_{n_0} flows will yield, in the same quasi-classical limit, a higher dimensional extension of the dispersionless Toda hierarchy [16].

7.4. Moyal versus Poisson algebraic hierarchies

Remarkably, unlike the Moyal algebraic hierarchies, these Poisson algebraic hierarchies are systems of “algebraic differential equations”—they consist of an infinite number of equations, but each equation contains only a finite number of terms. This is not the case for the Moyal algebraic hierarchy, because the Moyal bracket itself is an infinite power series of \hbar and contains arbitrarily high order derivatives of the unknown functions. First of all, Moyal algebraic deformations of the two heavenly equations themselves are non-algebraic in this sense.

A drawback is that the dressing operator method does not work directly within the Poisson algebraic hierarchies. The dressing operator method can be used only via the Moyal algebraic hierarchies. Twistor theory, which has been successful for selfdual gravity and the dispersionless KP and Toda hierarchies [10,12,16], will provide an alternative approach.

8. Conclusion—overview from the point of view of W -infinity algebras

Inspired by the philosophy of the large- N limit, we have constructed higher dimensional analogues of the KP hierarchy. Conceptually, the new hierarchies may be interpreted as the large- N limit of the N -component KP hierarchy. We, however, have avoided to deal with finite- N models and given a direct construction of such hierarchies. Fundamental constituents of the hierarchies are pseudo-differential operators with Moyal algebraic coefficients.

Actually, we have found two different types of hierarchy consisting, respectively, of commuting flows and noncommuting flows. The hierarchy of commuting flows resembles the ordinary and multi-component KP hierarchies, whereas the hierarchy of noncommuting flows includes Moyal algebraic deformations of selfdual gravity as a reduction.

Furthermore, both the commuting and noncommuting hierarchies have turned out to possess a quasi-classical limit, whose Lax formalism is based on Poisson brackets rather than Moyal brackets. In the quasi-classical limit, the commuting hierarchy gives a higher dimensional extension of the dis-

persionless KP hierarchy. The noncommuting hierarchy, meanwhile, includes ordinary selfdual gravity as a reduction.

It is instructive to review our results from the point of view of W -infinity algebras [17]. The following is a list of integrable systems that fall into our current scope:

- (0) ordinary KP hierarchy (KP),
- (1) N -component KP hierarchy (N -KP),
- (2) nonabelian KP hierarchies with Moyal algebraic coefficients (MAKP),
- (3) Moyal algebraic deformations of selfdual gravity (MASDG),
- (4) N -component AKNS–ZS hierarchy (N -AKNSZS).

These integrable systems are linked with each other as the following diagram indicates:

$$\begin{array}{ccccc}
 \text{KP} & \xrightarrow{N\text{-comp.}} & N\text{-KP} & \xrightarrow{N \rightarrow \infty} & \text{MAKP} \\
 & & \downarrow \hbar \partial_x \rightarrow \lambda & & \downarrow \hbar \partial_x \rightarrow \lambda \\
 & & N\text{-AKNSZS} & \xrightarrow{N \rightarrow \infty} & \text{MASDG}
 \end{array} \tag{8.1}$$

[Remark: More precisely, the large- N limit of N -AKNSZS gives a reduction of the commuting MAKP hierarchy rather than of the noncommuting MAKP hierarchy, hence the last line of the above diagram is somewhat inaccurate.]

The following list shows infinite dimensional Lie algebras associated with these integrable systems:

- (0) $W_\infty = \langle x^i (\hbar \partial_x)^j \rangle$ (ordinary W_∞ algebra [17]),
- (1) $W_\infty^N = \langle E_{\alpha\beta} x^i (\hbar \partial_x)^j \rangle$ (W_∞ algebra with inner symmetries [18]),
- (2) $W_\infty^\infty = \langle y^\alpha z^\beta x^i (\hbar \partial_x)^j \rangle$ (large- N limit of W_∞^N [18]),
- (3) $\mathcal{L}\text{Moyal}(\Sigma) = \langle y^\alpha z^\beta \lambda^j \rangle$ (loop algebra of Moyal algebra),
- (4) $\mathcal{L}\text{gl}(N) = \langle E^{\alpha\beta} \lambda^j \rangle$ (loop algebra of $\text{gl}(N)$),

where $\langle \dots \rangle$ shows generators; $E^{\alpha\beta}$ denote the standard generators of $\text{gl}(N)$,

$$(E^{\alpha\beta})_{ij} = \delta_{i\alpha} \delta_{j\beta}. \tag{8.2}$$

The first four of these algebras are of W -infinity type, whereas the last one is of Kac–Moody type. The above diagram of integrable systems simultaneously shows the interrelation of these Lie algebras:

$$\begin{array}{ccccc}
 W_\infty & \xrightarrow{N\text{-comp.}} & W_\infty^N & \xrightarrow{N \rightarrow \infty} & W_\infty^\infty \\
 & & \downarrow \hbar \partial_x \rightarrow \lambda & & \downarrow \hbar \partial_x \rightarrow \lambda \\
 & & \mathcal{L}\text{gl}(N) & \xrightarrow{N \rightarrow \infty} & \mathcal{L}\text{Moyal}(\Sigma)
 \end{array} \tag{8.3}$$

In the limit of $\hbar \rightarrow 0$, these W -infinity algebras are contracted to classical counterparts. In particular, $\mathcal{L}\text{Moyal}(\Sigma)$ and W_∞^∞ turn into Poisson algebras $\mathcal{L}\text{Poisson}(\Sigma)$ and $w_\infty^\infty \simeq \text{Poisson}(M)$, $\dim M = 4$. This is exactly what we have seen in the transition from Moyal algebraic hierarchies to Poisson algebraic hierarchies.

We conclude this paper with a list of issues to be pursued along the same line of approach. These issues will be discussed elsewhere.

(a) *Toda version of MAKP.* We noted in a previous section that a Toda version of MAKP will give a natural framework for dealing with the first heavenly equation of selfdual gravity. This should be a rather straightforward extension, simply replacing pseudo-differential operators by difference operators (with Moyal algebraic coefficients).

(b) *More detailed study of Poisson algebraic hierarchies.* The Poisson algebraic hierarchies will provide new material for the twistor theoretical approach to nonlinear integrable systems. Furthermore, as the dispersionless KP hierarchy is applied to Landau–Ginzburg models of topological strings [19], the higher dimensional hierarchies might be related to similar physical models.

(c) *Multi-component version of MAKP.* This extension is also straightforward, just redefining the pseudo-differential operators W , L , etc. to have $N \times N$ matrix valued coefficients with Moyal algebraic matrix elements. According to a preliminary analysis (we omit details), a hierarchy of Bogomolny-type [20] can be derived from this N -component version (N -MAKP) as a reduction. Thus the previous diagram can further be enlarged as:

$$\begin{array}{ccc} \text{MAKP} & \xrightarrow{N\text{-comp.}} & N\text{-MAKP} \\ & & \downarrow \text{reduction} \\ & & \text{Bogomolny} \end{array} \quad (8.4)$$

This reduction is however somewhat strange—constraints are imposed on both the dressing operator *and* the time variables. This interpretation of Bogomolny hierarchies resembles the “ D -module approach” of Sato et al. [21]. Furthermore, the fact that the time variables, too, have to be constrained is somewhat reminiscent of the “small phase space” of Landau–Ginzburg models [19].

(d) *Hierarchies with compactified Moyal algebras.* In the case of a torus, the Moyal algebra (or “quantum torus algebra”) has an invariant trace [22]. This will allow us to construct a higher dimensional tau function, because the usual trace of $\mathfrak{gl}(N)$ plays an implicit but substantial role in the theory of tau functions of multi-component KP hierarchies.

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